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MIXING RANK-ONE ACTIONS FOR INFINITE SUMS OF FINITE GROUPS

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ABSTRACT

Let G be a countable direct sum of finite groups. We construct an uncountable family of pairwise disjoint mixing (of any order) rank-one strictly ergodic free actions of G on a Cantor set. All of them possess the property of minimal self-joinings (of any order). Moreover, an example of rigid weakly mixing rank-one strictly ergodic free G-action is given.

0. Introduction and definitions

This paper was inspired by the following question of D. Rudolph:

QUESTION: *Which countable discrete amenable groups G have mixing (funny) rank-one free actions?*

Recall that a measure preserving action $T = (T_g)_{g\in G}$ of G on a standard probability space $(X, \mathfrak{B}, \mu)$ is called

— **mixing** if $\lim_{g\to\infty} \mu(A \cap T_g B) = \mu(A)\mu(B)$ for all $A, B \in \mathfrak{B}$,

— **mixing of order** l if for any $\epsilon > 0$ and $A_0, \dots, A_l \in \mathfrak{B}$, there exists a finite subset $K \subset G$ such that

$$|\mu(T_{g_0}A_0 \cap \dots \cap T_{g_l}A_l) - \mu(A_0)\cdots\mu(A_l)| < \epsilon$$

for each collection $g_0, \dots, g_l \in G$ with $g_i g_j^{-1} \notin K$ if $i \neq j$,

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— **weakly mixing** if the diagonal action $T \times T := (T_g \times T_g)_{g\in G}$ of G on the product space $(X \times X, \mathfrak{B} \otimes \mathfrak{B}, \mu \times \mu)$ is ergodic,
— **totally ergodic** if every co-finite subgroup in G acts ergodically,
— **rigid** if there exists a sequence $g_n \to \infty$ in G such that

$$\lim_{n\to\infty} \mu(A \cap T_{g_n} B) = \mu(A \cap B) \quad \text{for all } A, B \in \mathfrak{B}.$$

We say that T has **funny rank one** if there exist a sequence of measurable subsets $(A_n)_{n=1}^\infty$ in X and a sequence of finite subsets $(F_n)_{n=1}^\infty$ in G such that the subsets $T_g F_n$, $g \in F_n$, are pairwise disjoint for any n and

$$\lim_{n\to\infty} \min_{H \subset F_n} \mu\Big(B \triangle \bigsqcup_{g\in H} T_g A_n\Big) = 0 \quad \text{for every } B \in \mathfrak{B}.$$

If, moreover, $(F_n)_{n=1}^\infty$ is a subsequence of some 'natural' Følner sequence in G, we say that T has **rank one**. For instance, if $G = \mathbb{Z}^d$, this 'natural sequence' is just the sequence of cubes; if $G = \sum_{i=1}^\infty G_i$ with every G_i a finite group, the sequence $\sum_{i=1}^n G_i$ is 'natural', etc.

Up to now various examples of mixing rank-one actions were constructed for
— $G = \mathbb{Z}$ in [Or], [Ru], [Ad], [CrS], etc.,
— $G = \mathbb{Z}^2$ in [AdS],
— $G = \mathbb{R}$ in [Pr], [Fa],
— $G = \mathbb{R}^{d_1} \times \mathbb{Z}^{d_2}$ in [DaS].

We also mention two more constructions of rank-one actions for
— $G = \mathbb{Z} \oplus \bigoplus_{n=1}^\infty \mathbb{Z}/2\mathbb{Z}$ in [Ju], where it was claimed that the $\mathbb{Z}$-subaction is mixing but it was only shown that it is weakly mixing, and
— G is a countable Abelian group with a subgroup $\mathbb{Z}^d$ such that the quotient $G/\mathbb{Z}^d$ is locally finite in [Ma], where it was proved that a $\mathbb{Z}$-subaction is mixing and it was asked whether the whole action is mixing.

Notice that in all of these examples G is Abelian and has elements of infinite order. In contrast to that we provide a different class of groups for which the answer to the question of D. Rudolph is affirmative.

THEOREM 0.1: *Let $G = \bigoplus_{i=1}^\infty G_i$, where G_i is a non-trivial finite group for every i.*

(i) *There exist uncountably many pairwise disjoint (and hence pairwise non-isomorphic) mixing rank-one strictly ergodic actions of G on a Cantor set. Moreover, these actions are mixing of any order.*
(ii) *There exists a weakly mixing rigid (and hence non-mixing) rank-one strictly ergodic action of G on a Cantor set.*

Concerning (i), it is noteworthy that any mixing rank-one $\mathbb{Z}$-action is mixing of any order by [Ka] and [Ry] (see also an extension of that to actions of some Abelian groups with elements of infinite order in [JuY]). We do not know whether this fact holds for all mixing rank-one action of countable sums of finite groups.

To prove the theorem, we combine the original Ornstein's idea of 'random spacer' (in the cutting-and-stacking construction process) [Or] and the more recent (C, F)-construction developed in [Ju], [Da1], [Da2], [DaS1], [DaS2] to produce funny rank-one actions with various dynamical properties. However, unlike all of the known examples of (C, F)-actions, the actions in this paper are constructed without adding any spacer (cf. with [Ju], where all the spacers relate to $\mathbb{Z}$-subaction only). Instead of that on the n-th step we just cut the n-'column' into 'subcolumns' and then rotate each 'subcolumn' in a 'random way'. In the limit we obtain a topological G-action on a compact Cantor space.

Our next concern is to describe all ergodic self-joinings of the G-actions constructed in Theorem 0.1. Recall a couple of definitions.

Given two ergodic G-actions T and T' on $(X, \mathfrak{B}, \mu)$ and $(X', \mathfrak{B}', \mu')$ respectively, we denote by $J(T, T')$ the set of **joinings** of T and T', i.e. the set of $(T_g \times T'_g)_{g\in G}$-invariant measures on $\mathfrak{B} \otimes \mathfrak{B}'$ whose marginals on $\mathfrak{B}$ and $\mathfrak{B}'$ are μ and μ' respectively. The corresponding dynamical system $(X\times X', \mathfrak{B}\otimes\mathfrak{B}', \mu\times\mu')$ is also called a joining of T and T'. By $J^e(T, T') \subset J(T, T')$ we denote the subset of ergodic joinings of T and T' (it is never empty). In a similar way one can define the joininings $J(T_1, \dots, T_l)$ for any finite family $T_1, \dots, T_l$ of G-actions. If $J(T_1, \dots, T_l) = \{\mu_1 \times \cdots \times \mu_l\}$ then the family $T_1, \dots, T_l$ is called **disjoint**. If $T_1 = \cdots = T_l$ we speak about l-**fold self-joinings** of T_1 and use notation $J_l(T)$ for $J(\underbrace{T, \dots, T}_{l \text{ times}})$. For $g \in G$, we denote by $g^\bullet$ the conjugacy class of g. We also let

$$\mathrm{FC}(G) := \{g \in G \mid g^\bullet \text{ is finite}\}.$$

Clearly, $\mathrm{FC}(G)$ is a normal subgroup of G. If G is Abelian or G is a sum of finite groups then $\mathrm{FC}(G) = G$. For any $g \in \mathrm{FC}(G)$, we define a measure $\mu_{g^\bullet}$ on $(X \times X, \mathfrak{B} \otimes \mathfrak{B})$ by setting

$$\mu_{g^\bullet}(A \times B) := \frac{1}{\# g^\bullet} \sum_{h\in g^\bullet} \mu(A \cap T_h B).$$

It is easy to verify that $\mu_{g^\bullet}$ is a self-joining of T. Moreover, the map $(x, T_h^{-1}x) \mapsto (x, h)$ is an isomorphism of $(X \times X, \mu_{g^\bullet}, T \times T)$ onto $(X \times g^\bullet, \mu \times \nu, \widetilde{T})$, where

ν is the equidistribution on $g^\bullet$ and the G-action $\widetilde{T} = (\widetilde{T}_t)_{t\in G}$ is given by

$$\widetilde{T}_t(x,h) = (T_t x, tht^{-1}), \quad x \in X,\ h \in g^\bullet.$$

It follows that $\widetilde{T}$ (and hence the self-joining $\mu_{g^\bullet}$ of T) is ergodic if and only if the action $(T_t)_{t\in C(g)}$ is ergodic, where $C(g) = \{t \in G |\ tg = gt\}$ stands for the centralizer of g in G. Notice also that $C(g)$ is a co-finite subgroup of G because of $g \in \mathrm{FC}(G)$. Hence $\{\mu_{g^\bullet} |\ g \in \mathrm{FC}(G)\} \subset J_2^e(T)$ whenever T is totally ergodic.

Definition 0.2: If $J_2^e(T) \subset \{\mu_{g^\bullet} |\ g \in \mathrm{FC}(G)\} \cup \{\mu\times\mu\}$ then we say that T has **2-fold minimal self-joinings** (MSJ_2).

This definition extends naturally to higher order self-joinings as follows. Given $l \geq 1$ and $g \in G^{l+1}$, we denote by $g^{\bullet l}$ the orbit of g under the G-action on G^{l+1} by conjugations:

$$h \cdot (g_0, \dots, g_l) := (hg_0h^{-1}, \dots, hg_lh^{-1}).$$

Let P be a partition of $\{0, \dots, l\}$. For an atom $p \in P$, we denote by i_p the minimal element in p. We say that an element $g = (g_0, \dots, g_l) \in \mathrm{FC}(G)^{l+1}$ is **P-subordinated** if $g_{i_p} = 1_G$ for all $p \in P$. For any such g, we define a measure $\mu_{g^{\bullet l}}$ on $(X^{l+1}, \mathfrak{B}^{\otimes(l+1)})$ by setting

$$\mu_{g^{\bullet l}}(A_0 \times \cdots \times A_l) := \frac{1}{\# g^{\bullet l}} \sum_{(h_0,\dots,h_l)\in g^{\bullet l}} \prod_{p\in P} \mu\Big(\bigcap_{i\in p} T_{h_i} A_i\Big).$$

It is easy to verify that $\mu_{g^{\bullet l}}$ is an $(l+1)$-fold self-joining of T. Reasoning as above one can check that $\mu_{g^{\bullet l}}$ is ergodic whenever T is weakly mixing.

Definition 0.3: We say that T has **$(l+1)$-fold minimal self-joinings** (MSJ_{l+1}) if

$$J_{l+1}^e(T) \subset \{\mu_{g^{\bullet l}} |\ g \text{ is } P\text{-subordinated for a partition } P \text{ of } \{0, \dots, l\}\}.$$

If T has MSJ_l for any $l > 1$, we say that T has MSJ.

In case G is Abelian, these definitions agree with the—common now—definitions of MSJ_{l+1} and MSJ by A. del Junco and D. Rudolph [JuR] who considered self-joinings $\mu_{g^{\bullet l}}$ only when g belongs to the center of G^{l+1}. However, we find their definition somewhat restrictive for non-commutative groups since, for instance, countable sums of non-commutative finite groups can never have actions with MSJ_2 in their sense.

Now we record the second main result of this paper.

THEOREM 0.4: *The actions constructed in Theorem 0.1(i) all have MSJ.*

We notice that a part of the analysis from [Ru] can be carried over to the case of G-actions with MSJ. In this paper we only show that such actions have trivial *product centralizer*. Moreover, as follows from [Da3], every G-action with MSJ_2 is *effectively prime*, i.e. has no factors except for the obvious ones: the sub-σ-algebras of subsets fixed by finite normal subgroups in G. In particular, there exist no free factors.

We now briefly summarize the organization of the paper. In Section 1 we outline the (C, F)-construction of rank-one actions as it appeared in [Da1]. In Section 2, for any countable sum G of finite groups, we construct a (C, F)-action T of G which is mixing of any order. A rigid weakly mixing action of G also appears there. In Section 3 we demonstrate that T has MSJ. In Section 4 we show how to perturb the construction of T to obtain an uncountable family of pairwise disjoint mixing rank-one G-actions with MSJ. In the final Section 5 we discuss some implications of MSJ: trivial centralizer, trivial product centralizer and effective primality.

ACKNOWLEDGEMENT: The author thanks the referee for the useful suggestions that improved the paper. In particular, in the present proof of Theorem 0.4 we deduce MSJ_l from the l-fold mixing (as J. King does for $\mathbb{Z}$-actions in [Ki]). Our original proof (independent of multiple mixing) was longer and noticeably more complicated.

1. (C, F)-construction

In this section we recall the (C, F)-construction of rank-one actions.

From now on $G = \sum_{i=1}^{\infty} G_i$, where G_i is a non-trivial finite group for each $i \geq 1$. To construct a probability preserving (C, F)-action of G (see [Ju], [Da1], [DaS2]) we need to define two sequences $(F_n)_{n\geq 0}$ and $(C_n)_{n\geq 1}$ of finite subsets in G such that the following are satisfied:

(1.1) $$(F_n)_{n\geq 0} \text{ is a Folner sequence in } G, \quad F_0 = \{1_G\},$$

(1.2) $$F_n C_{n+1} \subset F_{n+1}, \quad C_{n+1} > 1,$$

(1.3) $$F_n c \cap F_n c' = \emptyset \quad \text{for all } c \neq c' \in C_{n+1},$$

(1.4) $$\lim_{n\to\infty} \frac{\#F_n}{\#C_1 \cdots \#C_n} < \infty.$$

Suppose that an increasing sequence of integers $0 < k_1 < k_2 < \cdots$ is given. Then we define $(F_n)_{n\geq 0}$ by setting $F_0 := \{1_G\}$ and $F_n := \sum_{i=1}^{k_n} G_i$ for $n \geq 1$.

Clearly, (1.1) is satisfied. Suppose now that we are also given a sequence of maps $s_n: H_n \to F_n$, where $H_0 := \sum_{i=1}^{k_1} G_i$ and $H_n := \sum_{i=k_n+1}^{k_{n+1}} G_i$ for $n \geq 1$. Then we define two sequences of maps $c_{n+1}, \phi_n: H_n \to F_{n+1}$ by setting $\phi_n(h) := (0, h)$ and $c_{n+1}(h) := (s_n(h), h)$. Finally, we let $C_{n+1} := c_{n+1}(H_n)$ for all $n \geq 0$. It is easy to verify that (1.2)–(1.4) are all fulfilled. Moreover, a stronger version of (1.2) holds:

$$F_n C_{n+1} = F_{n+1}. \tag{1.5}$$

We now put $X_n := F_n \times C_{n+1} \times C_{n+2} \times \cdots$ and define a map $i_n: X_n \to X_{n+1}$ by setting

$$i_n(f_n, d_{n+1}, d_{n+2}, \ldots) := (f_n d_{n+1}, d_{n+2}, \ldots).$$

Clearly, X_n is a compact Cantor space. It follows from (1.5) and (1.3) that i_n is well defined and it is a homeomorphism of X_n onto X_{n+1}. Denote by X the topological inductive limit of the sequence $(X_n, i_n)_{n=1}^{\infty}$. As a topological space X is canonically homeomorphic to any X_n and in the sequel we will often identify X with X_n suppressing the canonical identification maps. We need the structure of inductive limit to define the (C, F)-action T on X as follows. Given $g \in G$, consider any $n \geq 0$ such that $g \in F_n$. Every $x \in X$ can be written as an infinite sequence $x = (f_n, d_{n+1}, d_{n+2}, \ldots)$ with $f_n \in F_n$ and $d_m \in C_m$ for $m > n$ (i.e. we identify X with X_n). Now we put

$$T_g x := (g f_n, d_{n+1}, d_{n+2}, \ldots) \in X_n.$$

It is easy to verify that T_g is a well defined homeomorphism of X. Moreover, $T_g T_{g'} = T_{gg'}$, i.e. $T := (T_g)_{g \in G}$ is a topological action of G on X.

Definition 1.1: We call T **the (C, F)-action of G associated with the sequence $(k_n, s_{n-1})_{n=1}^{\infty}$**.

We list without proof several properties of T. They can be verified easily by the reader (see also [Da1]).

- T is a minimal uniquely ergodic (i.e. strictly ergodic) free action of G.
- Two points $x = (f_n, d_{n+1}, d_{n+2}, \ldots)$ and $x = (f'_n, d'_{n+1}, d'_{n+2}, \ldots) \in X_n$ are T-orbit equivalent if and only if $d_i = d'_i$ eventually (i.e. for all large enough i). Moreover, $x' = T_g x$ if and only if

$$g = \lim_{i \to \infty} f'_n d'_{n+1} \cdots d'_{n+i} d_{n+i}^{-1} \cdots d_{n+1}^{-1} f_n^{-1}.$$

- The only T-invariant probability measure μ on X is the product of the equidistributions on F_n and C_{n+i}, $i \in \mathbb{N}$ (if X is identified with X_n).

For each $A \subset F_n$, we let $[A]_n := \{x = (f_n, d_{n+1}, \dots) \in X_n | \ f_n \in A\}$ and call it an n-**cylinder**. The following holds:

$$
\begin{aligned}
&[A]_n \cap [B]_n = [A \cap B]_n, \quad \text{and} \quad [A]_n \cup [B]_n = [A \cup B]_n,\\
&[A]_n = \bigsqcup_{d \in C_{n+1}} [Ad]_{n+1},\\
&T_g[A]_n = [gA]_n \quad \text{if } g \in F_n,\\
&\mu([Ad]_{n+1}) = \frac{1}{\#C_{n+1}} \mu([A]_n) \quad \text{for any } d \in C_{n+1},\\
&\mu([A]_n) = \lambda_{F_n}(A),
\end{aligned}
$$

where λ_{F_n} is the normalized Haar measure on F_n. Moreover, for each measurable subset $B \subset X$,

$$\lim_{n\to\infty} \min_{A \subset F_n} \mu(B \triangle [A]_n) = 0. \tag{1.6}$$

Hence T has rank one.

2. Mixing (C, F)-actions

Our purpose in this section is to construct a rank-one action of G which is mixing of any order. This action will appear as a (C, F)-action associated with some specially selected sequence $(k_n, s_{n-1})_{n\geq 1}$. We first state several preliminary results.

Given finite sets A and B and a map $x \in A^B$, we denote by $\operatorname{dist} x$ or $\operatorname{dist}_{b\in B} x(b)$ the measure $(\#B)^{-1} \sum_{b \in B} \chi_{x(b)}$ on A. Here $\chi_{x(b)}$ stands for the probability supported at the point $x(b)$.

LEMMA 2.1: *Let A be a finite set and let λ be the equidistribution on A. Then for any $\epsilon > 0$ there exist $c > 0$ and $m \in \mathbb{N}$ such that for any finite set B with $\#B > m$,*

$$\lambda^B(\{x \in A^B | \ \|\operatorname{dist} x - \lambda\| > \epsilon\}) < e^{-c\#B}.$$

For the proof we refer to [Or] or [Ru]. We will also use the following combinatorial lemma.

LEMMA 2.2: *For any $l \in \mathbb{N}$, let $N_l := 3^{l(l-1)/2}$ and $\delta_l := 5^{-l(l-1)/2}$. Let H be a finite group. Then for any family $h_1, \dots, h_l$ of mutually different elements of H and any subset $B \subset H$ with $\#B > 3/\delta_l$, there exists a partition of B into subsets B_i, $1 \leq i \leq N_l$, such that the subsets $h_1 B_i, h_2 B_i, \dots, h_l B_i$ are mutually disjoint and $\#B_i \geq \delta_l \#B$ for any i.*

Proof: We leave to the reader the simplest case when $l = 2$. Hint: assume that $h_1 = 1_H$ and consider the partition of H into the right cosets by the cyclic group generated by h_2.

Suppose that we have already proved the assertion of the lemma for some l and we want to prove it for $l + 1$. Take any $h_1 \neq h_2 \neq \cdots \neq h_{l+1} \in H$ (in such a way we denote mutually different elements of H). Given a subset $B \subset H$ with $\#B > 3/\delta_l$, we first partition B into subsets B_i, $1 \leq i \leq N_l$, such that the subsets $h_2 B_i, h_3 B_i, \ldots, h_{l+1} B_i$ are mutually disjoint and $\#B_i \geq \delta_l \#B \geq 3 \cdot 5^l$. For every i, there exists a partition $B_i = \bigsqcup_{i_1=1}^{3} B_{i,i_1}$ such that $h_1 B_{i,i_1} \cap h_2 B_{i,i_1} = \emptyset$ and $\#B_{i,i_1} \geq 0.2\#B_i$, $1 \leq i_1 \leq 3$. Next, we partition every B_{i,i_1} into 3 subsets B_{i,i_1,i_2} such that $h_1 B_{i,i_1,i_2} \cap h_3 B_{i,i_1,i_2} = \emptyset$ and $\#B_{i,i_1,i_2} \geq 0.2\#B_{i,i_1}$, $1 \leq i_2 \leq 3$, and so on. Finally, we obtain a partition

$$B = \bigsqcup_{i=1}^{N_l} \bigsqcup_{i_1,\ldots,i_l=1}^{3} B_{i,i_1,\ldots,i_l}$$

which is as desired. ∎

Given a finite set A, a finite group H and elements $h_1, \ldots, h_l \in H$, we denote by $\pi_{h_1,\ldots,h_l}$ the map $A^H \to (A^l)^H$ given by

$$(\pi_{h_1,\ldots,h_l} x)(k) = (x(h_1 k), \ldots, x(h_l k)).$$

For $x \in A^H$, we define $x^* \in A^H$ by setting $x^*(h) := x(h^{-1})$, $h \in H$.

LEMMA 2.3: *Given $l \in \mathbb{N}$ and $\epsilon > 0$, there exists $m \in \mathbb{N}$ such that for any finite group H with $\#H > m$, one can find $s \in A^H$ such that*

$$\| \operatorname{dist} \pi_{h_1,\ldots,h_l} s - \lambda^l \| < \epsilon \quad \text{and} \quad \| \operatorname{dist} \pi_{h_1,\ldots,h_l} s^* - \lambda^l \| < \epsilon \tag{2.1}$$

for all $h_1 \neq h_2 \neq \cdots \neq h_l \in H$.

Proof: Take any finite group H and set

$$B_H := \bigcup_{h_1 \neq \cdots \neq h_l \in H} \{x \in A^H \mid \| \operatorname{dist} \pi_{h_1,\ldots,h_l} x - \lambda^l \| > \epsilon\}.$$

To prove the left hand side inequality in (2.1) it suffices to show that $\lambda^H(B_H) < 1$ whenever $\#H$ is large enough. Moreover, since the map $A^H \ni x \mapsto x^* \in A^H$ preserves the measure λ^H, the right hand side inequality in (2.1) will follow from the left hand side one if we prove that $\lambda^H(B_H) < 0.5$.

Fix $h_1 \neq \cdots \neq h_l \in H$ and apply Lemma 2.2 to partition H into subsets H_i, $1 \leq i \leq N_l$, such that

$$\#H_i \geq \delta_l \#H \quad \text{and} \tag{2.2}$$

$$\text{the subsets } h_1 H_i, \ldots, h_l H_i \text{ are mutually disjoint} \tag{2.3}$$

for every i. Denote by r_i: $(A^l)^H \to (A^l)^{H_i}$ the natural restriction map. Then we deduce from (2.3) that $r_i \circ \pi_{h_1,\cdots,h_l}$ maps λ^H onto $(\lambda^l)^{H_i}$. Since $\operatorname{dist} \pi_{h_1,\ldots,h_l} x = \sum_i (\#H_i/\#H) \cdot \operatorname{dist}(r_i \circ \pi_{h_1,\ldots,h_l})x$, it follows that

$$\begin{aligned} \lambda^H(\{x \in A^H | \, \| \operatorname{dist} \pi_{h_1,\ldots,h_l} x - \lambda^l \| > \epsilon\}) \\ &\leq \sum_i \lambda^H(\{x \in A^H | \, \| \operatorname{dist}(r_i \circ \pi_{h_1,\ldots,h_l})x - \lambda^l \| > \epsilon\}) \\ &= \sum_i (\lambda^l)^{H_i}(\{y \in (A^l)^{H_i} | \, \| \operatorname{dist} y - \lambda^l \| > \epsilon\}). \end{aligned}$$

By Lemma 2.2 and (2.2), there exists $c > 0$ such that if $\#H$ is large enough then the i-th term in the latter sum is less than $e^{-c\#H_i} < e^{-c\delta_l \#H}$. Hence

$$\lambda^H(B_H) \leq N_l \binom{\#H}{l} e^{-c\delta_l \#H}$$

and the assertion of the lemma follows. ∎

Now we are ready to define the sequence $(k_n, s_{n-1})_{n \geq 1}$. Fix a sequence of positive reals $\epsilon_n \to 0$. On the first step one can take arbitrary k_1 and s_0. Suppose now—on the n-th step—we already have k_n and s_{n-1} and we want to define k_{n+1} and s_n. For this, we apply Lemma 2.3 with $A := F_n$, $l := n$ and $\epsilon := \epsilon_n$ to find k_{n+1} large so that there exists $s_n \in A^{H_n}$ satisfying

$$\| \operatorname{dist} \pi_{h_1,\ldots,h_n} s_n - (\lambda_{F_n})^n \| < \epsilon_n \quad \text{for all } h_1 \neq \cdots \neq h_n \in H_n. \tag{2.4}$$

Recall that $H_n := \sum_{i=k_n+1}^{k_{n+1}} G_i$ and $F_n := \sum_{i=1}^{k_n} G_i$ for $n \geq 1$. Without loss of generality we may also assume that $k_{n+1} - k_n \geq n$ and hence $\sum_{n=1}^{\infty} (\#H_n)^{-1} < \infty$.

Denote by T the (C, F)-action of G on $(X, \mathfrak{B}, \mu)$ associated with $(k_n, s_{n-1})_{n=1}^{\infty}$.

THEOREM 2.4: *T is mixing of any order.*

Proof: (I) We first show that T is mixing (of order 1).

Recall that a sequence $g_n \to \infty$ in G is called **mixing for T** if

$$\lim_{n \to \infty} \mu(T_{g_n} B_1 \cap B_2) = \mu(B_1)\mu(B_2) \quad \text{for all } B_1, B_2 \in \mathfrak{B}.$$

Clearly, T is mixing if and only if any sequence going to infinity in G contains a mixing subsequence. Since every subsequence of a mixing sequence is mixing itself, to prove (I) it suffices to show that every sequence $(g_n)_{n=1}^{\infty}$ in G with $g_n \in F_{n+1} \setminus F_n$ for all n is mixing. Notice first that there exist (unique) $f_n \in F_n$ and $h_n \in H_n \setminus \{1\}$ with $g_n = f_n \phi_n(h_n)$. Fix any two subsets $A, B \subset F_n$. We notice that for each $h \in H_n$,

$$g_n A c_{n+1}(h) = f_n A s_n(h) \phi_n(h_n h) = f_n A s_n(h) s_n(h_n h)^{-1} c_{n+1}(h_n h)$$

and $f_n A s_n(h) s_n(h_n h)^{-1} \subset F_n$. Hence

$$\begin{aligned} \mu(T_{g_n}[A]_n \cap [B]_n) &= \sum_{h \in H_n} \mu(T_{g_n}[A c_{n+1}(h)]_{n+1} \cap [B]_n) \\ &= \sum_{h \in H_n} \mu([f_n A s_n(h) s_n(h_n h)^{-1} c_{n+1}(h_n h)]_{n+1} \cap [B]_n) \\ &= \sum_{h \in H_n} \mu([(f_n A s_n(h) s_n(h_n h)^{-1} \cap B) c_{n+1}(h_n h)]_{n+1}) \\ &= \frac{1}{\# H_n} \sum_{h \in H_n} \mu([f_n A s_n(h) s_n(h_n h)^{-1} \cap B]_n) \\ &= \frac{1}{\# H_n} \sum_{h \in H_n} \lambda_{F_n}(f_n A s_n(h) \cap B s_n(h_n h)). \end{aligned} \tag{2.5}$$

We define a map $r_{A,B}: F_n \times F_n \to \mathbb{R}$ by setting

$$r_{A,B}(g, g') := \lambda_{F_n}(f_n A g \cap B g').$$

Then it follows from (2.5) and (2.4) that

$$\begin{aligned} \mu(T_{g_n}[A]_n \cap [B]_n) &= \int_{F_n \times F_n} r_{A,B} d(\operatorname{dist} \pi_{1,h_n} s_n) \\ &= \int_{F_n \times F_n} r_{A,B} d\lambda_{F_n \times F_n} \pm \epsilon_n \\ &= \int_{F_n \times F_n} \lambda_{F_n}(f_n A g \cap B g') d\lambda_{F_n}(g) d\lambda_{F_n}(g') \pm \epsilon_n \\ &= \lambda_{F_n}(A) \lambda_{F_n}(B) \pm \epsilon_n \\ &= \mu([A]_n) \mu([B]_n) \pm \epsilon_n. \end{aligned}$$

Hence we have

$$\max_{A,B \subset F_n} |\mu(T_{g_n}[A]_n \cap [B]_n) - \mu([A]_n)\mu([B]_n)| < \epsilon_n. \tag{2.6}$$

This and (1.6) imply that the sequence $(g_n)_{n=1}^{\infty}$ is mixing.

(II) Now we fix $l > 1$ and prove that T is mixing of order l. To this end it is sufficient to show the following: given $l+1$ sequences $(g_{0,n})_{n=1}^{\infty}, \dots, (g_{l,n})_{n=1}^{\infty}$ in G such that $g_{i,n} \in F_{n+1}$ and $g_{i,n}g_{j,n}^{-1} \notin F_n$ whenever $i \neq j$,

$$\max_{A_0,\dots,A_l} |\mu(T_{g_{0,n}}[A_0]_n \cap \dots \cap T_{g_{l,n}}[A_l]_n) - \mu([A_0]_n)\cdots\mu([A_l]_n)| < \epsilon_n$$

for all $n > l$. Notice that for every $n \in \mathbb{N}$ and $0 \le j \le l$, there exist unique $f_{j,n} \in F_n$ and $h_{j,n} \in H_n$ with $g_{j,n} = f_{j,n}\phi_n(h_{j,n})$. Moreover, $h_{0,n} \neq h_{2,n} \dots \neq h_{1,n}$. Then slightly modifying the argument in (I), we compute

$$\begin{aligned} &\mu(T_{g_{0,n}}[A_0]_n \cap \dots \cap T_{g_{l,n}}[A_l]_n) \\ &= \int_{F_n^l} \lambda_{F_n}(f_{0,n}A_0g_0 \cap \dots \cap f_{l,n}A_lg_l) d(\lambda_{F_n})^{l+1}(g_0,\dots,g_l) \pm \epsilon_n \qquad (2.7) \\ &= \lambda_{F_n}(A_0)\cdots\lambda_{F_n}(A_l) \pm \epsilon_n = \mu([A_0]_n)\cdots\mu([A_l]_n) \pm \epsilon_n. \end{aligned}$$ ∎

To construct a weakly mixing rigid action of G we define another sequence $(\widetilde{k}_n, \widetilde{s}_{n-1})_{n\ge 1}$. When n is odd, we choose $\widetilde{k}_n$ and $\widetilde{s}_{n-1}$ to satisfy the following weaker version of (2.4):

$$\max_{1\neq h\in H_n} \|\operatorname{dist} \pi_{1,h}s_n - \lambda_{F_n} \times \lambda_{F_n}\| < \epsilon_n. \qquad (2.8)$$

When n is even, we just set $\widetilde{k}_n := \widetilde{k}_{n-1} + 1$ and $\widetilde{s}_n \equiv 1_G$. Denote by $\widetilde{T}$ the (C,F)-action of G on $(\widetilde{X}, \widetilde{\mathfrak{B}}, \widetilde{\mu})$ associated with $(\widetilde{k}_n, \widetilde{s}_{n-1})_{n=1}^{\infty}$.

THEOREM 2.5: *$\widetilde{T}$ is weakly mixing and rigid.*

Proof: Take any sequence $h_n \in H_{2n} \setminus \{1\}$. It follows from part (I) of the proof of Theorem 2.4 and (2.8) that the sequence $(\phi_{2n}(h_n))_{n=1}^{\infty}$ is mixing for $\widetilde{T}$. Clearly, it is also mixing for $\widetilde{T} \times \widetilde{T}$. Hence $\widetilde{T} \times \widetilde{T}$ is ergodic, i.e. $\widetilde{T}$ is weakly mixing.

Now take any sequence $h_n \in H_{2n+1} \setminus \{1\}$. Notice that (2.5) holds for any choice of $(k_n, s_{n-1})_{n\ge 1}$. Hence we deduce from (2.5) and the definition of $\widetilde{s}_{2n+1}$ that

$$\mu(\widetilde{T}_{\phi_{2n+1}(h_n)}[A]_{2n+1} \cap [B]_{2n+1}) = \lambda_{F_{2n+1}}(A \cap B) = \mu([A\cap B]_{2n+1})$$

for all subsets $A, B \subset F_{2n+1}$. This plus (1.6) yield

$$\lim_{n\to\infty} \mu(\widetilde{T}_{\phi_{2n+1}(h_n)}\widetilde{A} \cap \widetilde{B}) = \mu(\widetilde{A} \cap \widetilde{B})$$

for all $\widetilde{A}, \widetilde{B} \in \widetilde{\mathfrak{B}}$. This means that $\widetilde{T}$ is rigid. ∎

3. Self-joinings of T

This section is devoted entirely to the proof of the following theorem.

THEOREM 3.1: *The action T constructed in the previous section has MSJ.*

Proof: (I) We first show that T has MSJ_2. Since T is weakly mixing, we need to establish that

$$J_2^e(T) = \{\mu_{g^\bullet} \mid g \in G\} \cup \{\mu \times \mu\}.$$

Take any $\nu \in J_2^e(T)$. Let $\mathfrak{F}_n$ denote the sub-σ-algebra of $(T_g \times T_g)_{g\in F_n}$-invariant subsets. Then $\mathfrak{F}_1 \supset \mathfrak{F}_2 \supset \cdots$ and $\bigcap_n \mathfrak{F}_n = \{\emptyset, X \times X\} \pmod{\nu}$. Since there are only countably many cylinders, we deduce from the martingale convergence theorem that for ν-a.a. (x, x'),

$$E(\chi_{B\times B'}|\mathfrak{F}_{n-1})(x,x') = \frac{1}{\#F_{n-1}} \sum_{g\in F_{n-1}} \chi_{B\times B'}(T_g x, T_g x') \to \nu(B \times B') \tag{3.1}$$

as $n \to \infty$ for any pair of cylinders $B, B' \subset X$. Fix such a point (x, x'). It is called **generic** for $(T \times T, \nu)$. Given any $n > 0$, we can write x and x' as infinite sequences

$$x = (f_n, d_{n+1}, d_{n+2}, \ldots) \quad \text{and} \quad x' = (f'_n, d'_{n+1}, d'_{n+2}, \ldots)$$

with $f_n, f'_n \in F_n$ and $d_i, d'_i \in C_i$ for all $i > n$. Recall that $f_n := f_0 d_1 \cdots d_n$ and $f'_n := f'_0 d'_1 \cdots d'_n$. We set $t_n := f'_n f_n^{-1}$, $n > 0$. Fix a pair of cylinders, say m-cylinders, B and B'. If $n > m$ and $g \in F_n$ then $T_g x' = (g f'_n, d'_{n+1}, d'_{n+2}, \ldots)$. Hence $T_g x' \in B'$ if and only if $T_g T_{t_n} x \in B'$. Therefore

$$\chi_{B\times B'}(T_g x, T_g x') = \chi_{T_g^{-1}B\cap T_{t_n}^{-1}T_g^{-1}B'}(x).$$

Since x is generic for (T, μ), it follows that

$$\lim_{l\to\infty} \frac{1}{\#F_l} \sum_{a\in F_l} \chi_{T_g^{-1}B\cap T_{t_n}^{-1}T_g^{-1}B'}(T_a x) = \mu(T_g^{-1}B \cap T_{t_n}^{-1}T_g^{-1}B').$$

Therefore (3.1) yields

$$\lim_{n\to\infty} \frac{1}{\#F_{n-1}} \sum_{g\in F_{n-1}} \mu(T_g^{-1}B \cap T_{t_n}^{-1}T_g^{-1}B') = \nu(B \times B'). \tag{3.2}$$

Consider now two cases. If $t_n \notin F_{n-1}$ for infinitely many n, then passing to the limit in (3.2) along this subsequence and making use of (2.6) we obtain that $\mu(B)\mu(B') = \nu(B \times B')$. Hence $\mu \times \mu = \nu$. If, otherwise, there exists $N > 0$

such that $t_n \in F_{n-1}$, i.e. $d_n = d'_n$, for all $n > N$, then x and x' are T-orbit equivalent, $t_n = t_N$ and

$$\frac{1}{\#F_{n-1}} \sum_{g \in F_{n-1}} \mu(T_g^{-1} B \cap T_{t_n}^{-1} T_g^{-1} B') = \frac{1}{\#F_N} \sum_{g \in F_N} \mu(B \cap T_g T_{t_N}^{-1} T_g^{-1} B')$$
$$= \mu_{(t_N^{-1})_\bullet}(B \times B').$$

Passing to the limit in (3.1) we obtain that $\nu = \mu_{(t_N^{-1})_\bullet}$.

(II) Now we fix $l > 1$ and show that T has MSJ_{l+1}. Take any joining $\nu \in J^e_{l+1}(T)$ and fix a generic point $(x_0, \dots, x_l)$ for $(T \times \dots \times T, \nu)$. Define a partition P of $\{0, \dots, l\}$ by setting: i_1 and i_2 are in the same atom of P if x_{i_1} and x_{i_2} are T-orbit equivalent. As in (I), for any n, we can write

$$x_j = (f_{j,n-1}, d_{j,n}, d_{j,n+1}, \dots) \in X_{n-1}, \quad j = 0, \dots, l.$$

Suppose first that $\#P = l+1$, i.e. P is the finest possible. Then by the proof of (I), each 2-dimensional marginal of ν is $\mu \times \mu$. Since $\sum_{i=1}^\infty (\#C_i)^{-1} < \infty$ and $\mu = \lambda_{F_0} \times \lambda_{C_1} \times \lambda_{C_2} \times \cdots$, it follows from the Borel–Cantelli lemma that for ν-a.a. $(y_0, \dots, y_l) \in X^{l+1}$,

$$\exists N > 0 \text{ such that } y_{0,i} \neq y_{1,i} \neq \cdots \neq y_{l,i} \text{ whenever } i > N,$$

where $y_{j,i} \in C_i$ is the i-th coordinate of $y_j \in F_0 \times C_1 \times C_2 \times \cdots$. Hence without loss of generality we may assume that this condition is satisfied for $(x_0, \dots, x_l)$. Thus, if we set $t_{j,n} := f_{j,n} {f_{0,n}}^{-1} = f_{j,n-1} d_{j,n} {d_{0,n}}^{-1} {f_{0,n-1}}^{-1}$ then $t_{j,n} t_{i,n}^{-1} \notin F_{n-1}$ whenever $i \neq j$. Slightly modifying our reasoning in (I) and making use of (2.7) instead of (2.6) we now obtain

$$\begin{aligned}
\nu(B_0 \times \cdots \times B_l) &= \lim_{n \to \infty} \sum_{g \in F_{n-1}} \chi_{B_0 \times \cdots \times B_l}(T_g x_0, \dots, T_g x_l) \\
&= \lim_{n \to \infty} \sum_{g \in F_{n-1}} \chi_{B_0 \times \cdots \times B_l}(T_g x_0, T_g T_{t_{1,n}} x_0, \dots, T_g T_{t_{l,n}} x_0) \\
&= \lim_{n \to \infty} \sum_{g \in F_{n-1}} \mu(T_g B_0 \cap T_{t_{1,n}}^{-1} T_g B_1 \cap \ldots \cap T_{t_{l,n}}^{-1} T_g B_l) \\
&= \mu(B_0) \cdots \mu(B_l)
\end{aligned}$$

for any $(l+1)$-tuple of cylinders $B_0, \dots, B_l$. Hence $\nu = \mu \times \cdots \times \mu$.

Consider now the general case and put $t_{j,n} := f_{j,n} f_{i_p,n}^{-1}$ for each $j \in p$, $p \in P$. Recall that $i_p = \min_{j \in p} j$. Then

$$\chi_{B_0 \times \cdots \times B_l}(T_g x_0, \dots, T_g x_l) = \prod_{p \in P} \chi_{A_p}(x_{i_p}),$$

where $A_p := \bigcap_{j\in p} T_{t_{j,n}}^{-1} T_g^{-1} B_j$. Notice that the point $(x_{i_p})_{p\in P} \in X^{\{i_p|\, p\in P\}}$ is generic for $(T\times\cdots\times T(\#P \text{ times}), \kappa)$, where κ stands for the projection of ν onto $X^{\{i_p|p\in P\}}$. By the first part of (II), $\kappa = \mu\times\cdots\times\mu$ ($\#P$ times). Hence

$$\begin{aligned}\nu(B_0\times\cdots\times B_l) &= \lim_{n\to\infty}\frac{1}{\#F_{n-1}}\sum_{g\in F_{n-1}}\chi_{B_0\times\cdots\times B_l}(T_g x_0,\dots,T_g x_l)\\ &= \lim_{n\to\infty}\frac{1}{\#F_{n-1}}\sum_{g\in F_{n-1}}\prod_{p\in P}\mu(A_p).\end{aligned}$$

As in (I), a 'stabilization' property holds: there exists $M>0$ such that $t_{j,n} = t_{j,M}$ for all $n > M$. We now set $g := (t_{0,M}^{-1},\dots,t_{l,M}^{-1})$. Clearly, g is P-subordinated. Hence

$$\nu(B_0\times\cdots\times B_l) = \frac{1}{\#F_M}\sum_{g\in F_M}\prod_{p\in P}\mu\Big(\bigcap_{j\in p} T_g T_{t_{j,M}} T_g^{-1} B_j\Big) = \mu_{g\bullet\iota}(B_0\times\cdots\times B_l). \blacksquare$$

4. Uncountably many mixing actions with MSJ

In this section the proof of Theorems 0.1(i) and 0.4 will be completed. We first apply Lemma 2.3 to construct k_{n+1} and $s_n, \widehat{s}_n \in F_n^{H_n}$ in such a way that (2.4) is satisfied for both s_n and $\widehat{s}_n$ and, in addition,

$$\| \operatorname*{dist}_{h\in H_n}(s_n(hk), \widehat{s}_n(hk')) - \lambda_{F_n}\times\lambda_{F_n}\| < \epsilon_n \tag{4.1}$$

for all $k, k' \in H_n$. Next, given $\sigma\in\{0,1\}^{\mathbb{N}}$ and $n\in\mathbb{N}$, we define $s_n^\sigma\colon H_n\to F_n$ by setting

$$s_n^\sigma = \begin{cases} s_n & \text{if } \sigma(n)=0,\\ \widehat{s}_n & \text{if } \sigma(n)=1.\end{cases}$$

Now we denote by T^σ the (C,F)-action of G associated with $(k_n, s_{n-1}^\sigma)_{n=1}^\infty$. Let Σ be an uncountable subset of $\{0,1\}^{\mathbb{N}}$ such that for any $\sigma,\sigma'\in\Sigma$, the subset $\{n\in\mathbb{N}|\ \sigma(n)\neq\sigma'(n)\}$ is infinite.

THEOREM 4.1:

(i) *For any $\sigma\in\{0,1\}^{\mathbb{N}}$, the action T^σ is mixing and has MSJ.*

(ii) *If $\sigma,\sigma'\in\Sigma$ and $\sigma\neq\sigma'$ then T^σ and $T^{\sigma'}$ are disjoint.*

Proof: (i) follows from the proof of Theorem 3.1, since (2.4) is satisfied for s_n^σ for all $\sigma\in\{0,1\}^{\mathbb{N}}$ and $n\in\mathbb{N}$.

(ii) Let $\nu\in J^e(T^\sigma, T^{\sigma'})$. Take a generic point (x,x') for $(T^\sigma\times T^{\sigma'},\nu)$. Consider any n such that $\sigma(n)\neq\sigma'(n)$. Then we can write x and x' as infinite sequences $x = (f_n, d_{n+1}, d_{n+2}, \dots)$ and $x' = (f'_n, d'_{n+1}, d'_{n+2}, \dots)$ with $f_n, f'_n \in F_n$ and $d_m, d'_m \in C_m$ for all $m > n$. Take any $g \in F_{n+1}$. Then we have the following expansions:

$$g = a\phi_n(h), \quad d_{n+1} = s_n^\sigma(h_n)\phi_n(h_n) \quad \text{and} \quad d'_{n+1} = s_n^{\sigma'}(h'_n)\phi_n(h'_n)$$

for some uniquely determined $a \in F_n$ and $h, h_n, h'_n \in H_n$. Since

$$\begin{aligned} gf_nd_{n+1} &= af_ns_n^\sigma(h_n)s_n^\sigma(hh_n)^{-1}c_{n+1}(hh_n) \quad \text{and} \\ gf'_nd'_{n+1} &= af'_ns_n^{\sigma'}(h'_n)s_n^{\sigma'}(hh'_n)^{-1}c_{n+1}(hh'_n), \end{aligned}$$

the following holds for any pair of subsets $A, A' \subset F_n$:

$$\begin{aligned} &\frac{\#\{g \in F_{n+1} |\, (T_g^\sigma x, T_g^{\sigma'} x') \in [A]_n \times [A']_n\}}{\#F_{n+1}} \\ &= \frac{1}{\#F_n} \sum_{a \in F_n} \frac{\#\{h \in H_n |\, af_ns_n^\sigma(h_n)s_n^\sigma(hh_n)^{-1} \in A, af'_ns_n^{\sigma'}(h'_n)s_n^{\sigma'}(hh'_n)^{-1} \in A'\}}{\#H_n} \\ &= \frac{1}{\#F_n} \sum_{a \in F_n} \xi_n(A^{-1}af_ns_n^\sigma(h_n) \times A'^{-1}af'_ns_n^{\sigma'}(h'_n)), \end{aligned}$$

where $\xi_n := \text{dist}_{h \in H_n}(s_n^\sigma(hh_n), s_n^{\sigma'}(hh'_n))$. This and (4.1) yield

$$\begin{aligned} \frac{\#\{g \in F_{n+1} |\, (T_g^\sigma x, T_g^{\sigma'} x') \in [A]_n \times [A']_n\}}{\#F_{n+1}} &= \lambda_{F_n}(A)\lambda_{F_n}(A') \pm \epsilon_n \\ &= \mu([A]_n)\mu([A']_n) \pm \epsilon_n. \end{aligned} \tag{4.2}$$

Since (x, x') is generic for $(T^\sigma \times T^{\sigma'}, \nu)$ and (4.2) holds for infinitely many n, we deduce that $\nu = \mu \times \mu$. ∎

By refining the above argument the reader can strengthen Theorem 0.1(i) as follows: there exists an uncountable family of mixing (of any order) rank-one G-actions with MSJ such that any finite subfamily of it is disjoint.

5. On G-actions with MSJ

It follows immediately from Definition 0.2 that if T has MSJ_2 then the centralizer $C(T)$ of T is 'trivial', i.e. $C(T) = \{T_g |\, g \in C(G)\}$, where $C(G)$ denotes the center of G. Moreover, we will show that T has *trivial product centralizer* (as D. Rudolph did in [Ru] for $\mathbb{Z}$-actions).

Let $(X^l, \mathfrak{B}^{\otimes l}, \mu^l, T^{(l)})$ denote the l-fold Cartesian product of $(X, \mathfrak{B}, \mu, T)$. Given a permutation σ of $\{1, \dots, l\}$ and $g_1, \dots, g_n \in C(T)$, we define a transformation $U_{\sigma, g_1, \dots, g_l}$ of $(X^l, \mathfrak{B}^{\otimes l}, \mu^l, T^{(l)})$ by setting

$$U_{\sigma, g_1, \dots, g_l}(x_1, \dots, x_l) := (T_{g_1} x_{\sigma(1)}, \dots, T_{g_l} x_{\sigma(l)}).$$

Of course, $U_{\sigma,g_1,\dots,g_l} \in C(T^{(l)})$. We show that for the actions with MSJ, the converse also holds.

PROPOSITION 5.1: *If T has MSJ then, for any $l \in \mathbb{N}$, each element of $C(T^{(l)})$ equals to $U_{\sigma,g_1,\dots,g_l}$ for some permutation σ and elements $g_1,\dots,g_l \in C(G)$.*

Proof: Let $S \in C(T^{(l)})$. We define an ergodic 2-fold self-joining ν of $T^{(l)}$ by setting $\nu(A \times B) := \mu^l(A \cap S^{-1}B)$ for all $A, B \in \mathfrak{B}^{\otimes l}$. Notice that $\nu \in J_{2l}^e(T)$. Since T has MSJ_{2l}, there exists a partition P of $\{1,\dots,2l\}$ and a P-subordinated element $g = (g_1,\dots,g_{2l}) \in \mathrm{FC}(G)^{2l}$ such that

$$\nu(A_1 \times \cdots \times A_{2l}) = \frac{1}{\#g^{\bullet 2l}} \sum_{(h_1,\dots,h_{2l}) \in g^{\bullet 2l}} \prod_{p \in P} \mu\Big(\bigcap_{i \in p} T_{h_i} A_i\Big) \tag{5.1}$$

for all subsets $A_1,\dots,A_{2l} \in \mathfrak{B}$. Substituting at first $A_1 = \cdots = A_l = X$ and then $A_{l+1} = \cdots = A_{2l} = X$ in (5.1), we derive that $\#P = l$, $\#p = 2$ for all $p \in P$ and $\#g^{\bullet 2l} = 1$. Hence $g_1,\dots,g_{2l} \in C(G)$ and there exists a bijection σ of $\{1,\dots,l\}$ such that $P = \{\{i, \sigma(i) + l\} \mid i = 1,\dots,l\}$. Therefore it follows from (5.1) that

$$S^{-1}(A_{l+1} \times \cdots \times A_{2l}) = T_{g_{l+1}} A_{l+\sigma(1)} \times \cdots \times T_{g_{2l}} A_{l+\sigma(l)}. \qquad \blacksquare$$

As a simple corollary we derive that if T has MSJ then the G-actions $T, T^{(2)}, \dots$ and $T \times T \times \cdots$ are pairwise non-isomorphic.

After this paper was submitted the author introduced a companion to MSJ concept of *near simplicity* for actions of locally compact second countable groups [Da3]. As appeared, this concept is more general than the simplicity in the sense of A. del Junco and D. Rudolph [JuR] even for $\mathbb{Z}$-actions. For instance, there exist near simple transformations which are disjoint from all del Junco–Rudolph's simple ones. It is shown in [Da3] that an analogue of Veech's theorem on the structure of factors holds for this extended class of simple actions. In particular, if T has MSJ_2, then for every non-trivial factor $\mathfrak{F}$ of T there exists a compact normal subgroup K of G such that

$$\mathfrak{F} = \mathrm{Fix}\, K := \{A \in \mathfrak{B} \mid \mu(T_k A \triangle A) = 0 \text{ for all } k \in K\}.$$

Thus if T has MSJ_2 then T is *effectively prime*, i.e. T has no effective factors. (Recall that a G-action Q is called effective if $Q_g \neq \mathrm{Id}$ for each $g \neq 1_G$.)

References

[Ad] T. Adams, *Smorodinsky's conjecture on rank one systems*, Proceedings of the American Mathematical Society **126** (1998), 739–744.

[AdS] T. Adams and C. E. Silva, *$\mathbb{Z}^d$-staircase actions*, Ergodic Theory and Dynamical Systems **19** (1999), 837–850.

[CrS] D. Creutz and C. E. Silva, *Mixing on a class of rank-one transformations*, Ergodic Theory and Dynamical Systems **24** (2004), 407–440.

[Da1] A. I. Danilenko, *Funny rank-one weak mixing for nonsingular Abelian actions*, Israel Journal of Mathematics **121** (2001), 29–54.

[Da2] A. I. Danilenko, *Infinite rank one actions and nonsingular Chacon transformations*, Illinois Journal of Mathematics **48** (2004), 769–786.

[Da3] A. I. Danilenko, *On simplicity concepts for ergodic actions*, preprint.

[DaS1] A. I. Danilenko and C. E. Silva, *Multiple and polynomial recurrence for Abelian actions in infinite measure*, Journal of the London Mathematical Society (2) **69** (2004), 183–200.

[DaS2] A. I. Danilenko and C. E. Silva, *Mixing rank-one actions of locally compact Abelian groups*, preprint.

[Fa] B. Fayad, *Rank one and mixing differentiable flows*, Inventiones Mathematicae **160** (2005), 305–340.

[Ju] A. del Junco, *A simple map with no prime factors*, Israel Journal of Mathematics **104** (1998), 301–320.

[JuR] A. del Junco and D. Rudolph, *On ergodic actions whose self-joinings are graphs*, Ergodic Theory and Dynamical Systems **7** (1987), 531–557.

[JuY] A. del Junco and R. Yassawi, *Multiple mixing rank one group actions*, Canadian Journal of Mathematics **52** (2000), 332–347.

[Ka] S. A. Kalikow, *Twofold mixing implies threefold mixing for rank one transformations*, Ergodic Theory and Dynamical Systems **4** (1984), 237–259.

[Ki] J. L. King, *Joining-rank and the structure of finite rank mixing transformations*, Journal d'Analyse Mathématique **51** (1988), 182–227.

[Ma] B. Madore, *Rank-one group actions with simple mixing $\mathbb{Z}$-subactions*, New York Journal of Mathematics **10** (2004), 175–194.

[Or] D. S. Ornstein, *On the root problem in ergodic theory*, in *Proc. Sixth Berkley Symp. Math. Stat. Prob. (Univ. California, Berkeley, Calif., 1970/1971)*, Vol. II, University of California Press, Berkeley, Calif., 1972, pp. 347–356.

[Pr] A. Prikhodko, *Stochastic constructions of flows of rank one*, Matematicheskii Sbornik **192** (2001), 61–92.

[Ru] D. J. Rudolph, *An example of a measure preserving map with minimal self-joinings,* Journal d'Analyse Mathématique **35** (1979), 97–122.

[Ry] V. V. Ryzhikov, *Mixing, rank and minimal self-joining of actions with invariant measure,* Matematicheskii Sbornik **183** (1992), 133–160.